Dynamical transition from a quasi-one dimensional Bose-Einstein condensate to a Tonks-Girardeau gas

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We analyze in detail the expansion of a 1D Bose gas after removing the axial confinement. We show that during its one-dimensional expansion the density of the Bose gas does not follow a self-similar solution, but on the contrary, it asymptotically approaches a Tonks-Girardeau profile. Our analysis is based on a nonlinear Schrödinger equation with variable nonlinearity whose validity is discussed for the expansion problem, by comparing with an exact Bose-Fermi mapping for the case of an initial Tonks-Girardeau gas. For this case, the gas is shown to expand self-similarly, with a different similarity law compared to the one-dimensional Thomas-Fermi condensate.

During the last years, the achievement of Bose-Einstein condensation (BEC) [1] has generated an extraordinary interest in the physics of ultracold atomic gases. Among the topics related with the physics of ultracold gases, the issue of low-dimensionality is attracting a growing interest. The recent development of trapping and cooling techniques has enabled experimental realizations of low-dimensional gases both in one [2–4] and two [2,5–7] dimensions. For the case of ultracold low-dimensional dilute atomic gases, it has been theoretically predicted that 1D [8], 2D [9,10] and even very elongated but still dynamically 3D gases [11], should present an equilibrium BEC with a spatially fluctuating phase. Such quasicondensates have been recently observed by means of time of flight measurements [12].

The impenetrable 1D gas of bosons, the so-called Tonks-Girardeau (TG) gas, has recently deserved a special interest [8,13–16]. In order to accomplish this particular regime, rather strict conditions for the temperature, gas density, interaction potential, and trapping potential must be fulfilled [8,16]. These conditions can be achieved with currently available experimental techniques. Particularly important in this sense is the recent progress in loading 1D Bose gases in optical lattices [17] where the transversal confinement can reach 100kHz and the development of the Feshbach resonance techniques to modify the value of the s-wave scattering length [18]. Therefore, it is important to characterize the properties of the TG gas, and especially the intermediate regime between the quasi-1D BEC and the TG gas.

For the TG gas with a zero-range infinitely repulsive interatomic potential, the bosons acquire effectively a fermionic character and the mapping between bosonic and fermionic wavefunctions is exact, both for homogeneous [19] and trapped gases [14]. Interestingly, the homogeneous delta-interacting 1D bosonic gas under periodic boundary conditions is analytically solvable for any strength of the interactions, as shown by Lieb and Liniger (LL) [20]. There is unfortunately, to the best of our knowledge, no exact solution for arbitrary interaction strength in the case of trapped gases. An interesting approach was introduced in Ref. [21], where a hydrodynamic formalism was shown to reproduce the stationary properties of the TG gas. The approach of Ref. [21] should, however, be employed carefully since it significantly overestimates the coherence of the system [14]. Recently, the approach of Ref. [21] was extended to the case of finite interactions, by employing the LL model and local density approximation [16]. In Ref. [16] the density profile of the trapped gas was analyzed for regimes ranging from Thomas-Fermi (TF) profiles to TG. A different approach to the issue of finite interactions has been discussed in Ref. [23], where the intermediate regime is considered as a mixture of a BEC and a fermionized TG gas.

This Letter is devoted to the analysis of the 1D expansion of Bose gas. We employ the procedure of Ref. [16] to show that contrary to the case of a 1D Thomas-Fermi condensate, the expansion is not self-similar. In fact, any initial density profile will asymptotically evolve during its expansion towards a TG-like shape. Consequently the 1D expansion allows us to easily explore intermediate situations between the TF and TG regimes. Additionally, we discuss with the help of the Bose-Fermi (BF) mapping [14,15], the self-similar character of the expansion of an initial TG gas, which significantly differs from the self-similar expansion of a 1D TF cloud. Thus, the 1D expansion offers a way to clearly discern between TF and TG regimes, and in between. We justify the validity of the employed formalism for the expansion problem by comparing the hydrodynamical and the BF mapping results.

We consider in the following a dilute gas of N bosons confined in a very elongated harmonic trap with radial and axial frequencies ω_ρ and ω_z ($\omega_\rho \gg \omega_z$). If the interaction energy per particle is smaller than the zero-

point energy $\hbar\omega_\rho$ of the transversal trap, the system can be considered effectively as 1D. We first briefly review the formalism introduced in Ref. [16]. After approximating the interparticle interaction by a delta function, the Hamiltonian which describes the physics of the 1D gas becomes

$$\hat{H}_{1\mathrm{D}} = \hat{H}^0_{1\mathrm{D}} + \sum_{j=i}^{N} \frac{m\omega_z^2 z_i^2}{2} \tag{1}$$

with

$$\hat{H}^0_{1\mathrm{D}} = -\frac{\hbar^2}{2m}\sum_{j=1}^{N}\frac{\partial^2}{\partial z_j^2} + g_{1\mathrm{D}}\sum_{i=1}^{N}\sum_{j=i+1}^{N}\delta\left(z_i - z_j\right) \tag{2}$$

where m is the atomic mass and $g_{1\mathrm{D}} = -2\hbar^2/ma_{1\mathrm{D}}$. The one–dimensional scattering length is $a_{1\mathrm{D}} = (-a_\rho^2/2a)[1-\mathcal{C}(a/a_\rho)]$ [13] with a the three-dimensional scattering length, $a_\rho = \sqrt{2\hbar/m\omega_\rho}$ the oscillator length in the radial direction, and $\mathcal{C} = 1.4603\ldots$. As shown by Lieb and Liniger [20], $\hat{H}^0_{1\mathrm{D}}$ can be diagonalized by using Bethe Ansatz [22]. For the thermodynamic limit, a 1D gas at zero temperature with a given linear density n, is characterized by the energy per particle

$$\epsilon(n) = \frac{\hbar^2}{2m}n^2 e(\gamma(n)), \tag{3}$$

where $\gamma = 2/n|a_{1\mathrm{D}}|$. The function $e(\gamma)$ fulfills

$$e(\gamma) = \frac{\gamma^3}{\lambda^3(\gamma)}\int_{-1}^{1} g\left(x|\gamma\right)x^2 dx, \tag{4}$$

where $g\left(x|\gamma\right)$ and $\lambda(\gamma)$ are the solutions of the LL system of equations [20]

$$g\left(x|\gamma\right) = \frac{1}{2\pi} + \frac{1}{2\pi}\int_{-1}^{1}\frac{2\lambda(\gamma)}{\lambda^2(\gamma)+(y-x)^2}g\left(y|\gamma\right)dy \tag{5}$$

$$\lambda(\gamma) = \gamma\int_{-1}^{1} g\left(x|\gamma\right)dx. \tag{6}$$

We assume next that at each point z the gas is in local equilibrium, with local energy per particle provided by Eq. (3). Then, one can obtain the corresponding hydrodynamic equations for the density and the atomic velocity

$$\frac{\partial}{\partial t}n + \frac{\partial}{\partial z}(nv) = 0 \tag{7a}$$

$$\frac{\partial}{\partial t}v + v\frac{\partial}{\partial z}v = -\frac{1}{m}\frac{\partial}{\partial z}(\phi(n) + \frac{1}{2}m\omega_z^2 z^2). \tag{7b}$$

where

$$\phi(n) = \left(1 + n\frac{\partial}{\partial n}\right)\epsilon(n) \tag{8}$$

is the Gibbs free energy per particle. Inverting the corresponding Madelung transform, $\psi = \sqrt{n}\exp(iS)$, with $v = (\hbar/m)(\partial S/\partial z)$, one can reformulate Eqs. (7a) and (7b) in the form of a nonlinear Schrödinger equation (NLSE) of the form:

$$i\hbar\frac{\partial}{\partial t}\psi = \left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{1}{2}m\omega_z^2 z^2 + \phi(|\psi|^2)\right\}\psi. \tag{9}$$

Eq. (9) presents similar limitations as those of the NLSE of Ref. [21]. Its validity for the problem under consideration is discussed below. Note that for the case of $n|a_{1D}| \to \infty$, one obtains $\phi(n) = g_{1D}n$, retrieving the 1D Gross-Pitaevskii equation [24], whereas for the case $n|a_{1D}| \to 0$, one gets $\phi(n) = \pi^2\hbar^2 n^2/2m$, and the NLSE of Ref. [21] is recovered. The system has only one control parameter [16], namely $\eta = n^0_{TF}|a_{1D}|$, where $n^0_{TF} = [(9/64)N^2|a_{1D}|/a_z^4]^3$ is the TF density, with $a_z = \sqrt{\hbar/m\omega_z}$. The regime $\eta \gg 1$ corresponds to the TF limit, in which the stationary-state density profile has a parabolic form. On the other hand, the regime $\eta \ll 1$ corresponds to the TG regime, which is characterized by a stationary-state density profile with the form of a square root of a parabola.

We have employed Eqs. (3),(5),(6),(8) and (9) to simulate numerically the expansion of a 1D gas when the axial confinement is removed, i.e. $\omega_z = 0$ [25]. In our simulations we have employed a Crank-Nicholson method. Special care must be paid to the spatial and temporal integration steps, due to the long integration times needed, the velocities acquired during the expansion, and the larger nonlinearity in comparison to the case of the standard GPE.

In order to clearly understand the physics of the expansion dynamics, let us consider the case of a bosonic cloud which is initially TF-like ($n \sim (1 - x^2/R^2)$, with R the corresponding TF radius). This would be the case of an initial $\eta \gg 1$. During the course of the expansion, the cloud density decreases, following in the first stages a self-similar TF solution $n(z,t) = n(z/b(t), t=0)/b(t)$ with $\ddot{b} = \omega_z^2/b^2$ [26,27]. However, as the density decreases, the gas enters from the large $n|a_{1D}|$ regime into the low $n|a_{1D}|$ regime. As a consequence, the functional dependence of $\phi(n)$ changes throughout the whole cloud, and the expansion becomes no more TF self-similar (Fig. 1). When this happens, the density profile departs from a parabolic TF profile, and asymptotically evolves towards a square root of a parabola shape, i.e. the density profile becomes TG-like. Similar behavior is observed for any initial value of η.

In our simulations we have considered due to numerical limitations η values close to 1. We have analyzed in detail the dynamical transition from the initial density profile towards a TG shape, and observed that during the expansion the density profile presents at any time the form

$$n(z,t) = C(t)\left(1 - \left(\frac{z}{R(t)}\right)^2\right)^{s(t)} \tag{10}$$

where $R(t)$ is the radius of the cloud, and the exponent $s(t)$ takes the value $s(0) = 1$ for an initial TF gas. The normalization constant is of the form $C(t) = (N/\sqrt{\pi}R(t))(\Gamma(s(t)+3/2)/\Gamma(s(t)+1))$. The values of $R(t)$ and $s(t)$ are determined at any time by means of a nonlinear least-squares fitting algorithm. In order to check the validity of the fit, we have also considered the normalization constant as a fit parameter, and compared the obtained value with the expected value $C(t)$. The difference is less than 0.1%. As observed in Fig. 2 (for $\eta = 1$), the function $s(t)$ decreases monotonically in time, and it will asymptotically reach the value 0.5 The function $s(t)$ presents two clear time scales. It decreases fast during the first stages of the expansion (few axial trap periods), but the final convergence towards the TG profile is significantly slower. The latter is expected since the exact n^2 dependence of the functional $\phi(n)$ just appears asymptotically. We have analyzed the expansion for larger initial values of η, and observed a similar behavior, although for larger values of η the convergence towards $s = 0.5$ requires longer times (for $\eta = 3$ it should occur at around 350 axial trap periods). For practical purposes, if η is not sufficiently close to 1, only the first stage of the evolution of $s(t)$ will be observable, since for longer time scales the density will significantly decrease. For $\eta \gg 1$, the usual TF self-similar solution ($s(t) = const = 1$) is retrieved for any practical purposes. Note, however, that the transition into a TG profile during a 1D free expansion is in principle unavoidable for whatever initial condition, i.e. the s coefficient tends to 0.5. This is supported by the fact that the density profile fulfills at any time Eq. (10), and that the only self-similar solution supported by the NLSE (9) at very low densities is with $s = 0.5$.

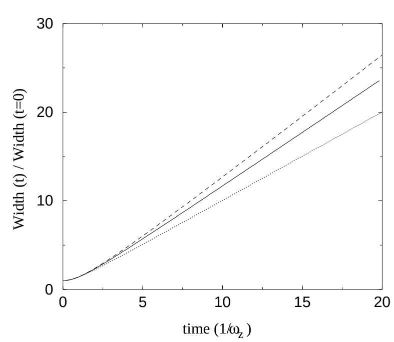

FIG. 1. Cloud width $\sqrt{\langle z^2 \rangle}$ as a function of time. The solid line is for $\eta = 1$, $\omega_r = 2\pi \times 20$kHz and $N = 200$ atoms ($\omega_z = 2\pi \times 1.8$Hz at $t = 0$). The dashed (dotted) lines are the self-similar 1D TF (TG) solutions.

Once shown that Eq. (9) predicts that during a free 1D expansion the density profile dynamically becomes TG-like, let us analyze the validity of the equation for the problem under consideration. As shown in Ref. [14], the hydrodynamical approach should be carefully employed, since it overestimates the coherence in the system. In order to check that Eq. (9) provides the right physical picture in our problem, we have calculated the free expansion of an initial TG gas using both the BF map, and the NLSE.

From the BF map one obtains that the dynamics of the density profile for an impenetrable gas of bosons is given by [14]

$$n(z,t) = \sum_{n=0}^{N} |\phi_n(z,t)|^2, \tag{11}$$

where $\phi_n(z,t)$ denotes the time-dependent wavefunction of the n-th eigenmode of the original axial harmonic oscillator. The expansion dynamics for each ϕ_n is obtained analytically by means of the corresponding Green function in free space

$$G(z-z',t) = -i\left(\frac{m}{2\pi\hbar i t}\right)^{1/2} e^{im(z-z')^2/2\hbar t}. \tag{12}$$

From Eq. (11) one obtains a self-similar solution of the form

$$n(z,t) = \frac{1}{\sqrt{1+\omega_z^2 t^2}} n\left(\frac{z}{\sqrt{1+\omega_z^2 t^2}}, t=0\right). \tag{13}$$

Note, that for times $t \gg 1/\omega_z$ the scaling coefficient $\sqrt{1+\omega_z^2 t^2}$ becomes $\omega_z t$, whereas for the case of a 1D TF self-similar solution the scaling coefficient becomes $\sqrt{2}\omega_z t$. Consequently, the expansion of an initial TG and TF gas is significantly different. This property could be employed to discern between the two regimes.

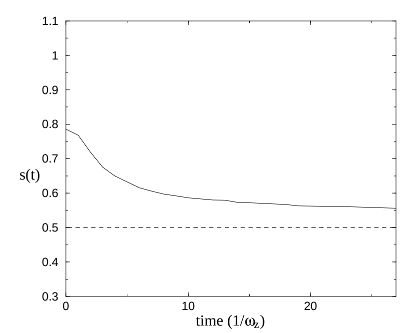

FIG. 2. Time evolution of the exponent $s(t)$ for $\eta = 1$, for the same parameters as in Fig. 1. The exponent decreases monotonically, and will asymptotically approach 0.5.

From the corresponding hydrodynamic equations (7a) and (7b), one can easily prove that the same self-similar solution (13) for the density is obtained from Eq. (9) in the limit of $n|a_{1D}| \to 0$, i.e. using the equation of Ref. [21]. Therefore, Eq. (9) accurately describes the expansion dynamics even for the extreme case of a TG gas, and it is thus expected to describe well the expansion for intermediate regimes between the TG and the TF limits, where the coherence is not yet completely lost.

In this Letter, we have studied the dynamical transition from a quasi-1D BEC into a TG gas during the expansion. Our analysis is based on a NLSE with variable nonlinearity, which generalizes for arbitrary interaction the extremal cases provided by the Gross-Pitaevskii equation ($n|a_{1D}| \to \infty$) and the equation of Ref. [21] ($n|a_{1D}| \to 0$). We have shown that even if the initial cloud possesses a TF profile, the density profile acquires asymptotically a TG shape. We have analyzed in detail this transition, and characterized the shape of the cloud in the intermediate stages. We have evaluated by means of a BF map the exact expansion dynamics of a TG gas and shown that the expansion is self-similar with a significantly different scaling law compared to a TF gas. We have additionally shown that the NLSE approach provides exactly the same self-similar solution as the BF map for the case of a TG gas, and it is therefore expected to describe well the expansion for any intermediate regime.

Let us additionally point out that the NLSE (9) also provides the excitation spectrum of the 1D Bose gas in intermediate regimes between TF and TG, by considering a small perturbation around the ground state solution $\psi_0(z)$ of Eq. (9) $\psi(z) = \psi_0(z) + \delta\psi(z)$, where $\delta\psi$ is given by $\delta\psi(z) = u(z)e^{-i\omega t} + v(z)^* e^{i\omega t}$. Inserting this Ansatz into Eq. (9) leads to the corresponding Bogoliubov-de Gennes equations

$$\mathcal{L}u(z) + n_0\phi'(n_0)v(z) = \hbar\omega u(z) \tag{14}$$

$$-\mathcal{L}v(z) - n_0\phi'(n_0)u(z) = \hbar\omega v(z) \tag{15}$$

where $n_0 = \psi_0^2$, $\phi' = d\phi/dn$, and $\mathcal{L} = -(\hbar^2/2m)(\partial^2/\partial z^2) + m\omega_z^2 z^2/2 + \phi(n_0) + n_0\phi'(n_0) - \mu$, with μ the chemical potential fixed by the normalization of n_0. Eqs. (14) and (15) describe the crossover from the TF to the TG regime for all excitation frequencies. In particular, we have obtained that these equations provide the same results as in Ref. [28] for the lowest compressional mode.

Summarizing, the 1D expansion dynamics constitutes an experimentally accessible tool to discern between the different interaction regimes in a 1D gas, and additionally provides a way to dynamically accomplish the TG gas. Unfortunately, the method employed in this Letter does not allow to analyze the fundamental problem of decoherence when entering the TG regime. The solution of this problem requires to extend the exact results of Refs. [14,15] to the case of inhomogeneous time dependent Bose gases with finite interactions, in which the BF mapping is not exact. Such analysis is beyond the scope of this Letter and it will be the subject of future investigations.

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